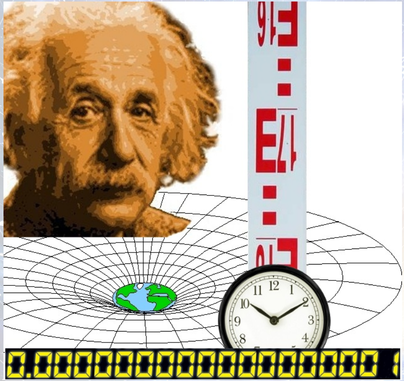
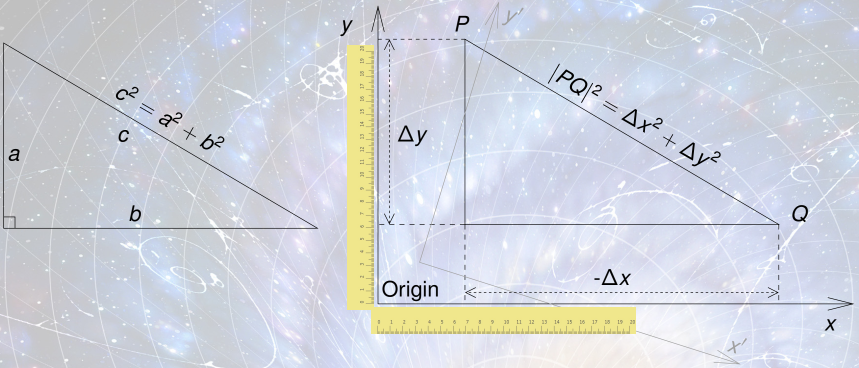
# Relativity and the future of vertical datums –

## about the geometry of space-time for geodesists

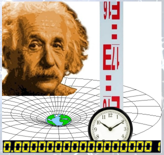
Martin Vermeer, Aalto University



# How the story starts: Pythagoras (1)




The Pythagoras theorem (left) and its version in rectangular co-ordinates (right): distance between $P$ and $Q$, $|PQ|$, is computable from two co-ordinate differences $\Delta x$ and $\Delta y$.



# Pythagoras (2)

The equation is

$$|PQ|^2 = \Delta x^2 + \Delta y^2 \left[ = (\Delta x')^2 + (\Delta y')^2 \text{, jne.} \right],$$

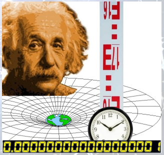
in which $\Delta x = x_P - x_Q$, $\Delta y = y_P - y_Q$ etc.

$|PQ|$ is an *invariant*: it doesn't depend on the co-ordinate frame used, $xy$ or $x'y'$. The mathematical form of Pythagoras, called the *metric*, describes the geometric behaviour of space.

Often, the differential form of the equation is used:

$$ds^2 = dx^2 + dy^2,$$

valid also on a curved surface if the distance between points $P$ and $Q$, $ds$, is small.



# Pythagoras (3)

In three-dimensional space, Pythagoras is

$$|PQ|^2 = \Delta x^2 + \Delta y^2 + \Delta z^2,$$

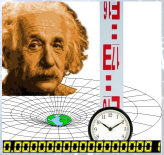
and in general *n*-dimensional space, its generalization would be

$$|PQ|^2 = \Sigma_{i=1}^{n} \left(\Delta x_i\right)^2 ,$$

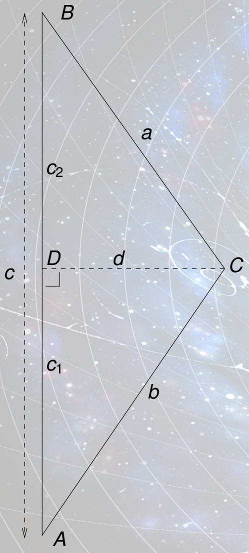
with *i* the “dimension counter”:

$$\begin{aligned} \Delta x_1 &= \Delta x \\ \Delta x_2 &= \Delta y \\ \Delta x_3 &= \Delta z \end{aligned}$$

etcetera. End of alphabet.



# Pythagoras and the triangle inequality



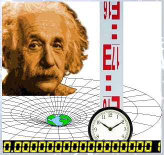

In the figure, Pythagoras tells us

$$
\begin{aligned}
a &= \sqrt{d^2 + c_2^2},\\
b &= \sqrt{d^2 + c_1^2},\\
c &= c_1 + c_2,
\end{aligned}
$$

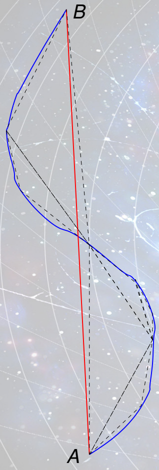
yielding

$$
\begin{aligned}
a + b &= \sqrt{c_2^2 + d^2} + \sqrt{c_1^2 + d^2} \geq \\
\geq \sqrt{c_2^2} + \sqrt{c_1^2} &= c_1 + c_2 = c.
\end{aligned}
$$

This is the *triangle inequality*: the shortest path from $A$ to $B$ is the straight path $ADB$. If $d > 0$, the path $ACB$ will always be longer.



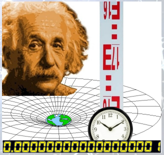
# The shortest distance and the geodesic




We may apply the triangle inequality repeatedly, i.e., *recursively* (dashed little triangles) to show that the non-straight curve (blue) is always longer than the straight one (red).

The shortest path is called the *geodesic*, also on a curved surface. Of course then it isn't really straight any more, just "as straight as possible".

E.g., on the surface of a sphere, a great circle is a geodesic. It is the path of an aircraft flying straight ahead, turning neither to port nor to starboard.



# Pythagoras in space-time (1)

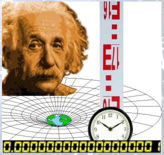
In space-time we have the co-ordinates time $t$, place $x$ (and $y$, and $z$). Points in space-time are called *events*. An example of events is the location in space-time of the same human being at different moments in his life.

Pythagoras is now a bit different, the general equation for the "interval" between events $P$ and $Q$ is

$$|PQ|^2 = \Delta t^2 - \mathfrak{c}^{-2} \left(\Delta x^2 + \Delta y^2 + \Delta z^2\right).$$

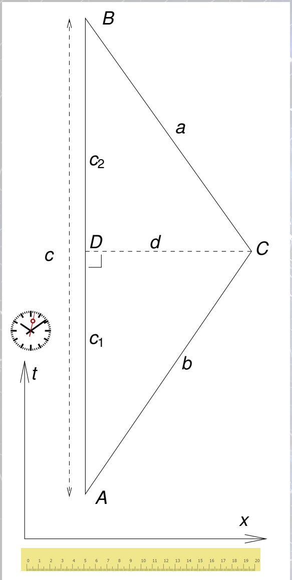
Note the minus sign and $\mathfrak{c}$, the speed of light. If $P$ and $Q$ are the same human being at different moments of his life, then this interval $|PQ|$ is the time elapsed between these moments.

If we use compatible or "natural" units (like years and light years), then $\mathfrak{c}$ drops out. Then



$$|PQ|^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2.$$

# Pythagoras in space-time (2)



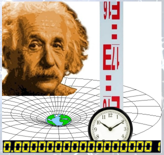

In the figure now

$$
\begin{aligned}
a^2 &= c_2^2 - d^2, \\
b^2 &= c_1^2 - d^2, \\
c &= c_1 + c_2,
\end{aligned}
$$

and

$$
\begin{aligned}
a + b &= \sqrt{c_2^2 - d^2} + \sqrt{c_1^2 - d^2} \leq \\
\leq \sqrt{c_1^2} + \sqrt{c_2^2} &= c_1 + c_2 = c.
\end{aligned}
$$

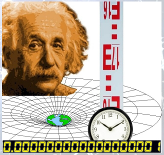
In other words, now the straight path (in time!) from event $A$ to event $B$ is the *longest* of all paths, not the shortest!



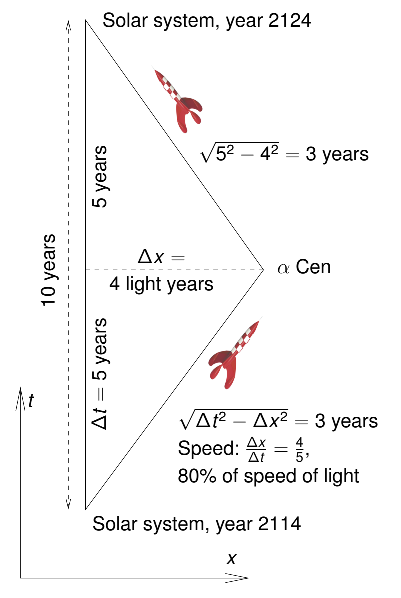
# Pythagoras in space-time (3)

The journey through space and time of an object, or a human, during their existence, is called their *world line*. The above result is, that the length of the world line between two events, the *proper time* spent on the journey, is maximal if the world line is *straight* – or more generally, a geodesic. More meandering journeys from one event to another are always *shorter*, i.e., they consume less proper time of the travelling human or object – or of a ticking clock.

The question to ask is whether the journey has *accelerations*. If, during the journey, one accelerates and brakes, less time is used than when the traveller is in free fall. From this follows the twin paradox: our intuition is informed by Pythagoras, the metric, for *space*, telling us that the detour is always the longer journey. In *space-time* it is just the opposite!

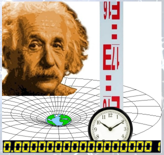


# The twin paradox




A concrete example. One brother travels to $\alpha$ Centauri, 4 light years away. He uses for the journey $3 + 3 = 6$ years of his own time, and upon return, in the year 2124, is four years younger than his twin brother who stayed home.

The travelling twin is the one who accelerates hard, brakes, accelerates and brakes again. The twin staying home, by comparison, undergoes hardly any accelerations at all.

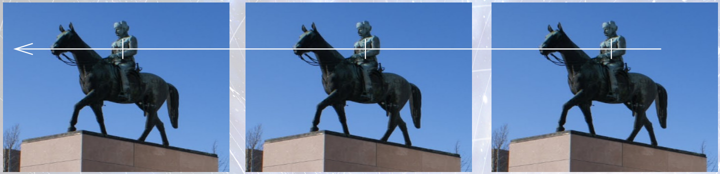


# The world line of the Mannerheim statue (1)

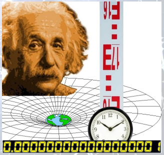


The Mannerheim statue stays in its place. . .

# The world line of the Mannerheim statue (2)



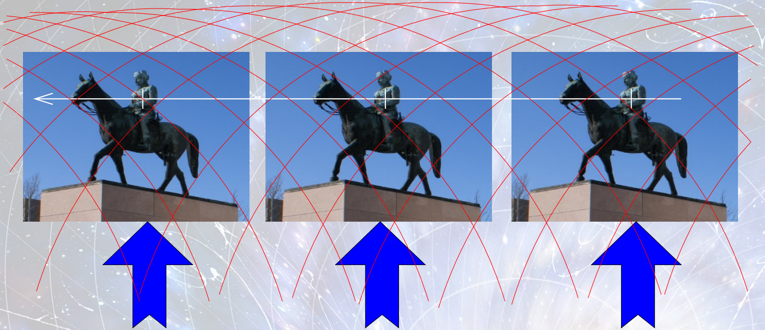
. . . so its path through *space* is very straight.



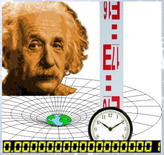
# The world line of the Mannerheim statue (3)

It is however not in a state of free fall: the Earth's surface pushes it continuously up, causing an acceleration of $9,8\ {}^{\mathrm{m}}/_{\mathrm{s}^2}$ in the upward direction!

# The world line of the Mannerheim statue (4)

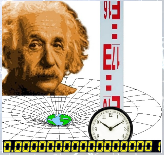


Therefore the world line of the statue *is not a geodesic*: it curves, in the curved space-time (red, conceptual art) softly upward, about five metres after a journey of one second (about 300 000 000 m) trough time.

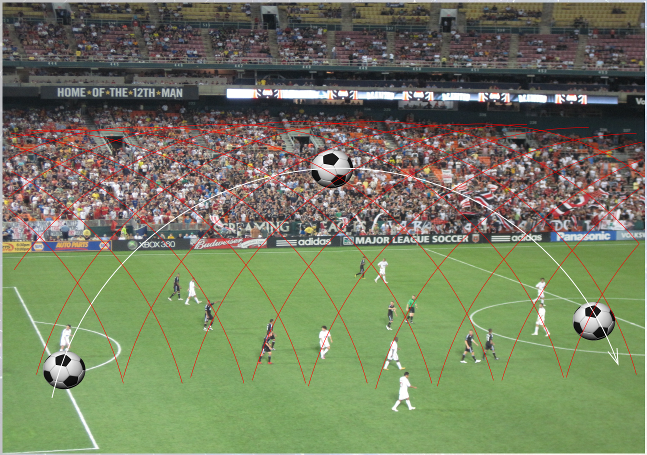


# The world line of a football (1)

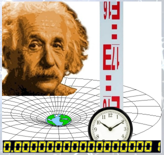

On the other hand a *football* is in free fall. And although its path in *space* is curved. . .



# The world line of a football (2)



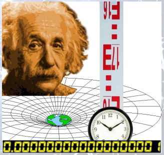

. . . its world line in the curved space-time surrounding the Earth is a *geodesic*, i.e., “straight”!



# How chronometric levelling works (1)

We have seen that the "proper time" of a clock, as it travels during its existence into the future, is the shorter, the more meandering its world line is. The world line of a clock resting on the Earth's surface is continually deviating from the straight line, i.e., the geodesic, under the influence of gravity – more precisely the resistance offered by the Earth's surface, see the above Mannerheim statue. Therefore it *loses time* compared to a reference clock that is, e.g., floating far away in space.

Theory tells us, that the time lost is directly proportional to the *geopotential of the location of measurement*, the gravity potential of the Earth. Gravity is the resultant of the Earth's gravitation (attraction) and the centrifugal force caused by the Earth's rotation. Thus one may, using precise clocks, measure potential differences between points, i.e., carry out *levelling*.

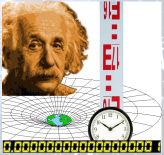


# How chronometric levelling works (2)

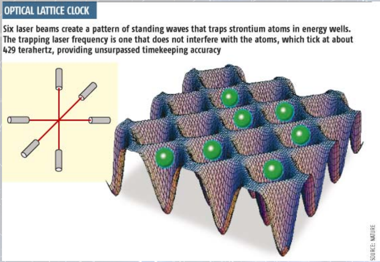
The equation is (Bjerhammar 1986, Vermeer 1983):

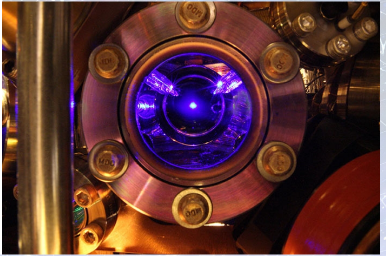
$$\frac{\Delta\tau}{\tau} = \frac{\Delta W}{\mathfrak{c}^2},$$

where $\tau$ is the time measured by a clock, $\Delta\tau$ the time difference between clocks, $\Delta W$ the potential difference between points, and $\mathfrak{c}$ the speed of light. We find quickly that a geopotential difference of 1 $\mathrm{m}^2/\mathrm{s}^2$, i.e., a height difference of 10 cm, can be measured with a relative clock precision of $10^{-17}$. In 1983 this sounded still pretty challenging, but in recent years, so-called optical lattice clocks have been developed, the precision of which is order-of-magnitude $10^{-18}$.



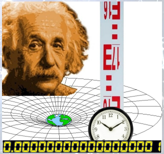
# Optical lattice clock



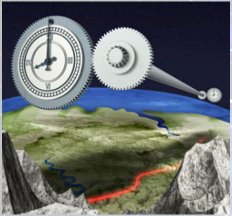



The technological novelty with optical lattice clocks is, that the wavelength used is *in the optical range* and not in the microwave range, like for more traditional atomic clocks. The more rapid oscillations allow for more precise time keeping.

Optical lattice – Nature – Physics World



# Time transfer by optic fibre



Measuring potential differences requires the comparison of clocks. Especially over longer distances, this is challenging on this level of precision.

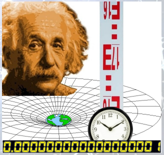
In Germany, the *Physikalisch-Technische Bundesanstalt* and the *Max-Planck-Institut für Quantenphysik* have developed a method in which one can use the pre-existing optic-fibre networks used by the Internet. Experiments (PTB/MPQ 2012) have shown this to be a working solution over distances of even 920 km. However, the signal amplifiers placed at regular distances along the cables must be replaced by specially manufactured ones.

# Applications

There are many applications for optical lattice clocks. Already in data communicatons, precise time keeping can be critical, and there, the new clocks will undoubtedly find use.

In theoretical physics the clocks will allow an even more precise testing of the general theory of relativity.

The most fascinating field of application will however be geodesy. One should note that, though the technology has in generally progressed, the most precise technique for measuring height differences continues to be precise levelling with traditional levelling instruments. The technique has many hard to control sources of error, part of them systematic. Perhaps the time has come for a new, very different levelling technology.

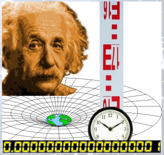


In Finland MIKES does optical lattice clock research.

# Geodetic infrastructure

As the technology matures, surely permanently operating heighting stations will be built interlinked with optic-fibre networks, in the same way as the already existing global network of permanently operating GNSS stations. In this way, not only a “zero order” height network is obtained, serving as a fundamental height reference, but monitoring of vertical motions of the Earth’s crust becomes possible.

A minor problem is posed by island points far from the optic-fibre network, as well as by linking different continents together: installing signal amplifiers into pre-existing ocean-floor cables isn’t straightforward. Perhaps for this purpose, synchronization using GNSS systems could be used, as originally proposed in Vermeer (1983). Such measurements would last for years.



# The German initiative

In Germany in 2014, a so-called Collaborative Research Centre (*Sonderforschungsbereich*) was started:

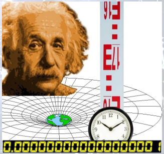
Geo-Q, “*Relativistic geodesy and gravimetry with quantum sensors*”

with federal funding €11 million for the fist four years. Developing chronometric levelling is part of this project.

DFG advert - Leibniz University Hannover - PTB text

Die Geodäten
arbeitsplatz-erde.de

http://www.arbeitsplatz-erde.de/



# Thank you for your interest!

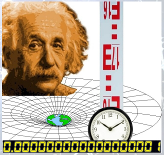
> "What good is a newborn baby?"
>
> – Benjamin Franklin, 1783, on aerial balloon experiments

**Literature**


Arne Bjerhammar (1975): *Discrete approaches to the solution of the boundary value problem in physical geodesy*. Boll. de geodesia e scienze affini.

Arne Bjerhammar (1986): *Relativistic Geodesy*. NOAA Technical Report NOS 1 18 NGS 36. https://www.ngs.noaa.gov/PUBS_LIB/RelativisticGeodesy_TR_NOS118_NGS36.pdf

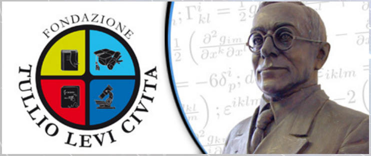
Martin Vermeer (1983): *Chronometric levelling*. Report 83:2 of the Finnish Geodetic Institute, Helsinki. ISBN 951-711-087-1, ISNN 0355-1962

# Epilogue: Einstein's long shadow







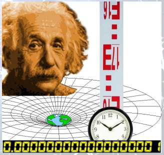

**Fermat** discovered the principle according to which light travels between two points along the *fastest* path.

**Gauss** discovered, simultaneously with János Bolyai and Nikolai Lobachevski, non-Euclidean geometry, and developed the theory of curved spaces.

**Hamilton** generalized the Fermat principle to apply to the motion of objects ▷ *Hamiltonian mechanics*. He disn't yet grasp why this was even possible. . .

**De Broglie** did grasp this: *also matter is a wave motion* (and conversely, light is made up of photons) ▷ *quantum theory*, particle–wave dualism.

Also the *geodesics* of relativity theory are paths like those of Hamilton, or Fermat. The are related to the concept of the *absolute derivative* in curved space-time, **Levi-Civita**'s wonderful idea.



To this could still be added **Pythagoras, Riemann, Maxwell**, . . .